Indecomposable coverings*

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Dedicated to the memory of László Fejes Tóth

Abstract. We prove that for every $k > 1$, there exist k-fold coverings of the plane (1) with strips, (2) with axis-parallel rectangles, and (3) with homothets of any fixed concave quadrilateral, that cannot be decomposed into two coverings. We also construct, for every $k > 1$, a set of points P and a family of disks $\mathcal{D}$ in the plane, each containing at least k elements of P, such that no matter how we color the points of P with two colors, there exists a disk $D \in \mathcal{D}$, all of whose points are of the same color.

1 Multiple arrangements: background and motivation

The notion of multiple packings and coverings was introduced independently by Davenport and László Fejes Tóth. Given a system $\mathcal{S}$ of subsets of an underlying set X, we say that they form a *k-fold packing (covering)* if every point of X belongs to *at most (at least)* k members of $\mathcal{S}$. A 1-fold packing (covering) is simply called a *packing (covering)*. Clearly, the union of k packings (coverings) is always a k-fold packing (covering). Today there is a vast literature on this subject [FTG83], [FTK93].

Many results are concerned with the determination of the maximum density $\delta^k(C)$ of a k-fold packing (minimum density $\theta^k(C)$ of a k-fold covering) with congruent copies of a fixed convex body C. The same question was studied for multiple *lattice packings (coverings)*, giving rise to the parameter $\delta^k_L(C)$ $(\theta^k_L(C))$. Throughout this paper, it is always assumed that the geometric arrangements, packings, and coverings under consideration are *locally finite*, that

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is, any bounded region intersects only finitely many members of the arrrangement.

Because of the strongly combinatorial flavor of the definitions, it is not surprising that combinatorial methods have played an important role in these investigations. For instance, Erdős and Rogers [ER62] used the "probabilistic method" to show that $\mathbf{R}^d$ can be covered with congruent copies (actually, with translates) of a convex body so that no point is covered more than $e(d \ln d + \ln\ln d + 4d)$ times (see [PA95], and [FuK05] for another combinatorial proof based on Lovász' Local Lemma). Note that this easily implies that there exist positive constants θ_d, δ_d, depending only on d, such that

$$k \leq \theta^k(C) \leq k\theta(C) \leq k\theta_d,$$

$$k\delta_d \leq k\delta(C) \leq \delta^k(C) \leq k.$$

Here $\delta(C)$ and $\theta(C)$ are shorthands for $\delta^1(C)$ and $\theta^1(C)$).

To establish almost tight density bounds, at least for lattice arrangements, it would be sufficient to show that any k-fold packing (covering) splits into roughly k packings (coverings), or into about k/l disjoint l-fold packings (coverings) for some $l < k$. The initial results were promising. Blundon [Bl57] and Heppes [He59] proved that for unit disks $C = B^2$, we have

$$\theta_L^2(C) = 2\theta_L(C), \quad \delta_L^k(C) = k\delta_L(C) \ \textit{for}\ k \leq 4,$$

and these results were extended to arbitrary centrally symmetric convex bodies in the plane by Dumir and Hans-Gill [DuH72] and by G. Fejes Tóth [FTG77], [FTG84]. In fact, there was a simple reason for this phenomenon: It turned out that every 3-fold lattice packing of the plane can be decomposed into 3 packings, and every 4-fold lattice packing into *two* 2-fold ones. This simple scheme breaks down for larger values of k. As k tends to infinity, Cohn [Co76] and Bolle [Bo89] proved that

$$\lim_{k\to\infty} \frac{\theta_L^k(C)}{k} = \lim_{k\to\infty} \frac{\theta^k(C)}{k} = 1 \leq \theta(C),$$

$$\lim_{k\to\infty} \frac{\delta_L^k(C)}{k} = \lim_{k\to\infty} \frac{\delta^k(C)}{k} = 1 \geq \delta(C).$$

For convex bodies C with a "smooth" boundary, the inequalities on the right-hand side are strict [Sch61], [Fl78].

The situation becomes slightly more complicated if we do not restrict our attention to *lattice* arrangements. In reply to a question raised by László Fejes Tóth, the senior author noted [P80] that any 2-fold packing of homothetic copies of a plane convex body splits into 4 packings. Furthermore, any k-fold packing $\mathcal{C}$ with not too "elongated" convex sets splits into at most $9\lambda k$ packings, where

$$\lambda := \max_{C\in\mathcal{C}} \frac{(\text{circumradius}(C))^2\pi}{\text{area}(C)}.$$

(Here the constant factor 9λ can be improved. See also [Ko04].)

One would expect that similar results hold for coverings rather than packings. However, in this respect we face considerable difficulties. For any k, it is easy to construct a k-fold covering of the plane with not too elongated convex sets (of different shapes but of roughly the same size) that cannot be decomposed even to *two* coverings [P80]. The problem is far from being trivial even for coverings with congruent disks. In an unpublished manuscript, P. Mani-Levitska and the (then junior and now) senior author have shown that every 33-fold covering of the plane with congruent disks splits into two coverings [MP87]. Another positive result was established in [P86].

Theorem 1.1. *For any centrally symmetric convex polygon P, there exists a constant $k = k(P)$ such that every k-fold covering of the plane with translates of P can be decomposed into two coverings.*

At first glance, one may believe that approximating a disk by centrally symmetric polygons, the last theorem implies that any sufficiently thick covering with congruent disks is decomposable. The trouble is that, as we approximate a disk with polygons P, the value $k(P)$ tends to infinity. Nevertheless, it follows from Theorem 1.1 that if $k = k(\varepsilon)$ is sufficiently large, then any k-fold covering with disks of radius 1 splits into a covering and an "almost covering" in the sense that it becomes a covering if we replace each of its members by a concentric disk whose radius is $1 + \varepsilon$.

Recently, Tardos and Tóth [TaT06] have managed to extend Theorem 1.1 to any (not necessarily centrally symmetric) convex polygon P. Here the assumption that P is convex cannot be dropped.

Surprisingly, the analogous decomposition result is false for multiple coverings with balls in *three* and higher dimensions.

Theorem 1.2. [MP87] *For any k, there exists a k-fold covering of $\mathbf{R}^3$ with unit balls that cannot be decomposed into two coverings.*

Somewhat paradoxically, it is the very heavily covered points that create problems. Pach [P80], [AS00] (p. 68) noticed that by the Lovász Local Lemma we obtain

Theorem 1.3. [AS00] *Any k-fold covering of $\mathbf{R}^3$ with unit balls, no $c2^{k/3}$ of which have a point in common, can be decomposed into two coverings. (Here c is a positive constant.)*

Similar theorems hold in $\mathbf{R}^d$ ($d > 3$), except that the value $2^{k/3}$ must be replaced by $2^{k/d}$.

2 Cover-decomposable families: statement of results

These questions can be reformulated in a slightly more general combinatorial setting. **Definition 2.1.** *A family $\mathcal{F}$ of sets in $\mathbf{R}^d$ is called* cover-decomposable *if there exists a positive integer $k = k(\mathcal{F})$ such that any k-fold covering of $\mathbf{R}^d$ with members from $\mathcal{F}$ can be decomposed into two coverings.*

In particular, Theorem 1.1 above can be rephrased as follows. The family consisting of all translates of a given centrally symmetric convex polygon in the plane is cover-decomposable. Theorem 1.2 states that the translates of a unit ball is 3-space is not cover-decomposible. These results are valid for both *open* and *closed* polygons and balls.

Note that Theorem 1.1 has an equivalent "dual" form. Given a system $\mathcal{S}$ of translates of P, let $C(\mathcal{S})$ denote the set of centers of all members of $\mathcal{S}$. Clearly, $\mathcal{S}$ forms a k-fold covering of the plane if and only if every translate of P contains at least k elements of $C(\mathcal{S})$. Recall that, by assumption, $\mathcal{S}$ is a locally finite arrangement. Therefore, any bounded region contains only *finitely many* points of $C(\mathcal{S})$. We call such a point set *locally finite*.

The fact that the family of translates of P is cover-decomposable can be expressed by saying that there exists a positive integer k satisfying the following condition: any locally finite set C of points in the plane such that $|P' \cap C| \geq k$ for all translates P' of P can be partitioned into two disjoint subsets C_1 and C_2 with

$$|C_1 \cap P'| \neq \emptyset \text{ and } |C_2 \cap P'| \neq \emptyset \text{ for every translate } P' \text{ of } P.$$

We can think of C_1 and C_2 as "color classes."

This latter condition, in turn, can be reformulated as follows. Let $H(C)$ denote the (infinite) hypergraph whose vertex set is C and whose (hyper)edges are precisely those subsets of C that can be obtained by taking the intersection of C by a translate of P. By assumption, every hyperedge of $H(C)$ is of size at least k. The fact that C can be split into two color classes C_1 and C_2 with the above properties is equivalent to saying that $H(C)$ is *two-colorable*.

Definition 2.2. *A hypergraph is* two-colorable *if its vertices can be colored by two colors such that no edge is monochromatic.*

A hypergraph is called two-edge-colorable *if its edges can be colored by two colors such that every vertex is contained in edges of both colors.*

Obviously, a hypergraph H is two-edge-colorable if and only if its *dual hypergraph* H^* is two-colorable. (By definition, the vertex set and the edge set of H^* are the edge set and the vertex set of H, respectively, with the containment relation reversed.)

Summarizing, Theorem 1.1 can be rephrased in two equivalent forms. For any centrally symmetric convex polygon P in the plane, there is a $k = k(P)$ such that

1. any k-fold covering of $\mathbf{R}^2$ with translates of P (regarded as an infinite hypergraph on the vertex set $\mathbf{R}^2$) is two-edge-colorable;
2. for any locally finite set of points $C \subset \mathbf{R}^2$ with the property that each translate of P covers at least k elements of C, the hypergraph $H(C)$ whose edges are the intersections of C with all translates of P is two-colorable.

Clearly, the above two statements are also equivalent for translates of *any* set P, that is, we do not have to assume here that P is a polygon or that it is convex or connected. However, if instead of *translates*, we consider congruent, similar, or homothetic copies of P, then assertions 1 and 2 do not necessarily remain equivalent.

The aim of this paper is to give various geometric constructions showing that certain families of sets in the plane are not cover-decomposable.

Let T_k denote a rooted k-ary tree of depth $k-1$. That is, T_k has $1+k+k^2+k^3+\ldots+k^{k-1}=\frac{k^k-1}{k-1}$ vertices. The only vertex at level 0 is the root v_0. For $0 \le i < k-1$, each vertex at level i has precisely k children. The k^{k-1} vertices at level $k-1$ are all leaves.

Definition 2.3. *For any rooted tree T, let $H(T)$ denote the hypergraph on the vertex set $V(T)$, whose hyperedges are all sets of the following two types:*
1. Sibling *hyperedges: for each vertex $v \in V(T)$ that is not a leaf, take the set $S(v)$ of all children of v;*
2. Descendent *hyperedges: for each leaf $v \in V(T)$, take all vertices along the unique path from the root to v.*

Obviously, $H_k =: H(T_k)$ is a k-uniform hypergraph with the following property. No matter how we color the vertices of H_k by two colors, red and blue, say, at least one of the edges will be monochromatic. In other words, H_k is not two-colorable. Indeed, assume without loss of generality that the root v_0 is red. The children of the root form a sibling hyperedge $S(v_0)$. If all points of $S(v_0)$ are blue, we are done. Otherwise, pick a red point $v_1 \in S(v_0)$. Similarly, there is nothing to prove if all points of $S(v_1)$ are blue. Otherwise, there is a red point $v_2 \in S(v_1)$. Proceeding like this, we either find a sibling hyperedge $S(v_i)$, all of whose elements are blue, or we construct a red descendent hyperedge $\{v_0, v_1, \ldots, v_{k-1}\}$.

Definition 2.4. *Given any hypergraph H, a* planar realization *of H is defined as a pair $(P, \mathcal{S})$, where P is a set of points in the plane and $\mathcal{S}$ is a system of sets in the plane such that the hypergraph obtained by taking the intersections of the members of $\mathcal{S}$ with P is isomorphic to H.*

A realization of the dual hypergraph of H is called a dual realization *of H.*

In the sequel, we show that for any rooted tree T, the hypergraph $H(T)$ defined above has both a planar and a dual realization, in which the members of $\mathcal{S}$ are open strips (Lemmas 3.1–4.1). In particular, the hypergraph $H_k = H(T_k)$ permits such realizations for every positive k. These results easily imply the following

Theorem 2.5. *The family of open strips in the plane is not cover-decomposable.*

Indeed, fix a positive integer k, and assume that we have shown that $H_k = H(T_k)$ has a dual realization with strips (see Lemma 4.1). This means that the set of vertices of T_k can be represented by a collection $\mathcal{S}$ of strips, and the set of (sibling and descendent) hyperedges by a point set $P \subset \mathbf{R}^2$ whose every element is covered by the corresponding k strips. Recall that H_k is not two-colorable, hence its dual hypergraph H_k^* is not two-edge-colorable. In other words, no matter how we color the strips in $\mathcal{S}$ with two colors, at least one point in P will be covered only by strips of the same color. Add now to $\mathcal{S}$ all open strips that do not contain any element of P. Clearly, the resulting (infinite) family of strips, $\mathcal{S}'$, forms a k-fold covering of the plane, and it does not split into two coverings. This proves Theorem 2.5.

In fact, a "degenerate" version of Theorem 2.5 is also true, in which strips are replaced by straight-lines (that is, by "strips of width zero").

Theorem 2.6. *The family of straight lines in the plane is not cover-decomposable.*

We prove this theorem in Section 4. It implies the following generalization of Theorem 2.5: The family of open strips of *unit width* in the plane is not cover-decomposable.

Lemma 5.1 was originally established in [MP87]. For completeness, here we include a somewhat simpler proof (see Section 5). Lemma 5.1 easily implies that, for any $d \geq 3$, the family of open unit balls in $\mathbf{R}^d$ is not cover-decomposable, for any $d \geq 3$ (Theorem 1.2).

In Section 6, we show that the hypergraph $H_k = H(T_k)$ permits a dual realization in the plane with axis-parallel rectangles, for every positive k (Lemma 6.1). This implies, in exactly the same way as outlined in the paragraph below Theorem 2.5, that the following theorem is true.

Theorem 2.7. *The family of axis-parallel open rectangles in the plane is not cover-decomposable.*

We cannot decide whether H_k permits a planar realization. However, it can be shown [CPST06] that a sufficiently large randomly and uniformly selected point set P in the unit square, say, with large probability has the following property. No matter how we color the points of P with two colors, there is an axis-parallel rectangle containing at least k elements of P, all of the same color.

Recall that the family of translates of any convex polygon Q is cover-decomposable (see Theorem 1.1 and [TaT06]). The next result shows that this certainly does not hold for some *concave* polygons Q.

Theorem 2.8. *The family of all translates of a given (open) concave quadrilateral is not cover-decomposable.*

The proofs presented in the next five sections also yield that Theorems 2.5, 2.7, and 2.8 remain true for *closed* strips, rectangles, and quadrilaterals. Most arguments follow the same general inductional scheme, but the subtleties require separate treatment.

3 Planar realization with strips

A *strip* is an open set S in the plane, bounded by two parallel lines. The counterclockwise angle α $\;(-\frac{\pi}{2} < \alpha \leq \frac{\pi}{2})$ from the x-axis to these lines is called the *direction* or *slope* of S.

Lemma 3.1. *For any rooted tree T, the hypergraph $H(T)$ permits a planar realization with strips. That is, there is a set of points P and a set of strips $\mathcal{S}$*

in the plane such that the hypergraph on the vertex set P whose hyperedges are the sets $S \cap P$ $(S \in \mathcal{S})$ is isomorphic to $H(T)$.

Proof: We prove the lemma by induction on the number of vertices of T. The statement is trivial if T has only one vertex. Suppose that T has n vertices and that the statement has been proved for all rooted trees with fewer vertices. Let v_0 be the root of T, and let $v_0v_1 \dots v_m$ be a path of maximum length starting at v_0. Let $U = \{u_1, u_2, \dots u_k\}$ be the set of children of v_{m-1}. Each member of U is a leaf of T, and one of them is v_m. Delete the members of U from T, and let T' denote the resulting rooted tree. Clearly, v_{m-1} is a leaf of T'. By the induction hypothesis, there is a planar realization $(P, \mathcal{S})$ of $H(T')$ with open strips. We can assume without loss of generality that no element of P lies on the boundary of any strip in $\mathcal{S}$, otherwise we could slightly decrease the widths of some strips without changing the containment relation.

Let $S \in \mathcal{S}$ be the strip representing the descendent hyperedge $\{v_0, v_1, \dots, v_{m-1}\}$, i.e., a strip that contains precisely the points corresponding to these vertices of T'. (See Definition 2.3.) Rotating S through very small angles, the resulting strips $S^1, S^2, \dots, S^k$ contain the same points of P as S does. Moreover, we can make sure that the new strips are not parallel to each other or to any old strip. Hence, we can choose a line ℓ, not passing through any element of P, such that $S^1, S^2, \dots, S^k$ intersect ℓ in pairwise disjoint intervals that are also disjoint from all members of $\mathcal{S}$. For each i, $1 \le i \le k$, pick a point p^i in $\ell \cap S^i$, and add these points to P. Replace S in $\mathcal{S}$ by the strips $S^1, S^2, \dots, S^k$, and add another member to $\mathcal{S}$: a very narrow strip $\bar{S}$ around ℓ, which contains all p^i, but no other point of P.

In this way, we obtain a planar realization of $H(T)$, where $p_1, p_2, \dots, p_k$ represent the vertices (leaves) $u_1, u_2, \dots u_k \in V(T)$, the strip $\bar{S}$ represents the sibling hyperedge $U = \{u_1, u_2, \dots u_k\}$ of $H(T)$, while $S^1, S^2, \dots, S^k$ represent the descendent hyperedges, corresponding to the paths from v_0 to $u_1, u_2, \dots u_k$, respectively. $\square$

A hypergraph is *k-uniform* if all of its hyperedges have precisely k vertices.

Corollary 3.2. *For any $k \ge 2$, there exists a k-uniform hypergraph which is not two-colorable and which permits a planar realization by open strips.* $\square$

4 Dual realization with strips: Proofs of Theorems 2.5 and 2.6

Recall that a *dual* realization of a hypergraph H is a planar realization of its dual H^*. That is, given a tree T, a dual realization of $H(T)$ is a pair $(P, \mathcal{S})$, where P is a set of points in the plane representing the (sibling and descendent) hyperedges of $H(T)$, and $\mathcal{S}$ is a system of regions representing the vertices of T such that a region $S \in \mathcal{S}$ covers a point $p \in P$ if and only if the vertex corresponding to S is contained in the hyperedge corresponding to p.

Lemma 4.1. *For any rooted tree T, the hypergraph $H(T)$ permits a dual realization with strips.*

Proof: Most of our proof is identical to the proof of Lemma 3.1. We establish the statement by induction on the number of vertices of T. The statement is trivial if T has only one vertex. Suppose that T has n vertices and that the statement has been proved for all rooted trees with fewer than n vertices. Let v_0 be the root of T, and let $v_0 v_1 \dots v_m$ be a path of maximum length starting at v_0. Let $U = \{u_1, u_2, \dots u_k\}$ denote the set of children of v_{m-1}. Clearly, each element of U is a leaf of T, one of them is v_m, and U is a sibling hyperedge of $H(T)$. Let T' denote the tree obtained by deleting from T all elements of U. The vertex v_{m-1} is then a leaf of T'.

By the induction hypothesis, $H(T')$ permits a dual realization $(P, \mathcal{S})$ with open strips. We can assume without loss of generality that no element of P lies on the boundary of any strip in $\mathcal{S}$, otherwise we could slightly decrease the widths of some strips without changing the containment relation.

Let $p \in P$ be the point corresponding to the descendent hyperedge $\{v_0, v_1, \dots, v_{m-1}\}$ of $H(T')$. Let $p_1, p_2, \dots, p_k$ be distinct points so close to p that they are contained in exactly the same strips from $\mathcal{S}$ as p (namely, in the ones corresponding to $v_0, v_1, \dots, v_{m-1}$). The point p_i will correspond to the descendent hyperedge of T containing v_i. Choose a point q such that all lines $p_i q$ for $1 \le i \le k$ are distinct and they do not pass through any element of P. This point will correspond to the sibling hyperedge $\{u_1, \dots, u_k\}$ of T. For $1 \le i \le k$, let S^i be an open strip around the line $p_i q$ that is narrow enough so that it does not contain any element of P or any point p_j with $j \ne i$. This strip represents the vertex u_i of T.

Add $S^1, S^2, \dots, S^k$ to $\mathcal{S}$. Delete p from P, and add $p_1, \dots, p_k$, and q. The resulting configuration is a dual realization of $H(T)$ with open strips, so we are done. □

Proof of Theorem 2.6: Let C_k^n be a $k \times k \times \ldots \times k$ piece of the n-dimensional integer grid, that is,

$$C_k^n = \{(x_1, x_2, \dots, x_n) : x_i \in \{0, 1, \dots, k-1\}\}.$$

A *k-line* is a set of k collinear points of C_k^n. Denote by H_k^n the k-uniform hypergraph on the vertex set C_k^n, whose hyperedges are the k-lines. The following statement is a direct consequence of the Hales-Jewett theorem.

Lemma 4.2. [HaJ63] *The hypergraph H_k^n is not two-colorable.*

Our goal is to construct an indecomposable covering of the plane by (continuously many) straight lines such that every point is covered at least k times. Project C_k^n to a "generic" plane so that no two elements of C_k^n are mapped into the same point and no three noncollinear points become collinear.

Applying a duality transformation, we obtain a family $\mathcal{L}$ of k^n lines and a set P of so-called *k-points*, dual to the k-lines, such that each k-point belongs to precisely k members of $\mathcal{L}$. It follows from Lemma 4.2 that for any two-coloring of the members of $\mathcal{L}$, there is a k-point $p \in P$ such that all lines passing through p are of the same color.

It remains to extend the family $\mathcal{L}$ into a k-fold covering of the whole plane with lines without destroying the last property. This can be achieved by simply adding to $\mathcal{L}$ all straight lines that do not pass through any point in P. $\Box$

5 Planar realization with disks

In this section, all *disks* are assumed to be open. A pair $(P, \mathcal{D})$ consisting of a point set P and a system of disks $\mathcal{D}$ in the plane is said to be in *general position*, if no element of P lies on the boundary of a disk $D \in \mathcal{D}$, no two members of $\mathcal{D}$ are tangent to each other, and no three circles bounding members of $\mathcal{D}$ pass through the same point.

In order to facilitate the induction, we prove a slightly stronger lemma than what we need.

Lemma 5.1. *For any rooted tree T, the hypergraph $H(T)$ permits a planar realization $(P, \mathcal{D})$ with disks in general position such that every disk $D \in \mathcal{D}$ has a point on its boundary that does not belong to the closure of any other disk $D' \in \mathcal{D}$.*

Proof: By induction on the number of vertices of T. The statement is trivial if T has only one vertex. Suppose that T has n vertices and that the statement has already been proved for all rooted trees with fewer than n vertices. Let v_0 denote the root of T, and let $v_0 v_1 \dots v_m$ be a path of maximum length starting at v_0. Let $U = \{u_1, u_2, \dots u_k\}$ be the set of children of v_{m-1}. Each element of U is a leaf of T, and one of them is v_m. Remove all elements of U from T, and let T' denote the resulting rooted tree. Clearly, v_{m-1} is a leaf of T'. By the induction hypothesis, $H(T')$ permits a planar realization $(P, \mathcal{D})$ with disks satisfying the conditions in the lemma.

Let D denote the disk representing the descendent hyperedge $\{v_0, v_1, \dots, v_{m-1}\}$ of $H(T')$. Let v be a point on the boundary of D, which does not belong to the closure of any other disk $D' \in \mathcal{D}$. Choose a small neighborhood $N(v, \varepsilon)$ of v, which still disjoint from any disk $D' \in \mathcal{D}$ other than D.

To obtain a planar realization of $H(T)$, we have to add k new points to P that will represent the vertices $u_1, u_2, \dots u_k \in V(T)$, and replace D by k new disks that will represent the descending hyperedges of $H(T)$, corresponding to the paths connecting the root to $u_1, u_2, \dots u_k$. We also add a disk representing the sibling hyperedge $U = \{u_1, u_2, \dots u_k\}$ of $H(T)$. This can be achieved, as follows.

Let ℓ denote the straight line connecting the center of D to v, and let w be the point on ℓ, outside of D, at distance $\varepsilon/2$ from v. Let $D(1), D(2), \dots, D(k)$ be k disks obtained from D by rotating it about the point w through very small angles, so that $D(i) \cap P = D \cap P$ holds for any $1 \le i \le k$. Further, let D' denote the disk of radius $\varepsilon/2$, centered at w. Then $D(i)$ and D are tangent to each other; let $p(i)$ denote their point of tangency $(1 \le i \le k)$. Add the points

$p(1), p(2), \ldots, p(k)$ to P; they will represent $u_1, u_2, \ldots, u_k \in V(T)$, respectively. Remove D from $\mathcal{D}$, and replace it by the disks $D(1), D(2), \ldots, D(k)$ and D'.

Now we are almost done: the new pair $(P, \mathcal{D})$ is almost a planar realization of $H(T)$, with the disk D' representing the sibling hyperedge $\{u_1, u_2, \ldots u_k\}$ of $H(T)$. The only problem is that the points $p(i)$ lie on the boundaries of $D(i)$ and D', rather than in their interiors. This can be easily fixed by increasing the radii of the disks $D(i) \quad (1 \le i \le k)$ and D' by a very small positive number $\delta < \varepsilon/2$, so that the enlarged D' contains $p(1), p(2), \ldots, p(k)$, but no other points in P.

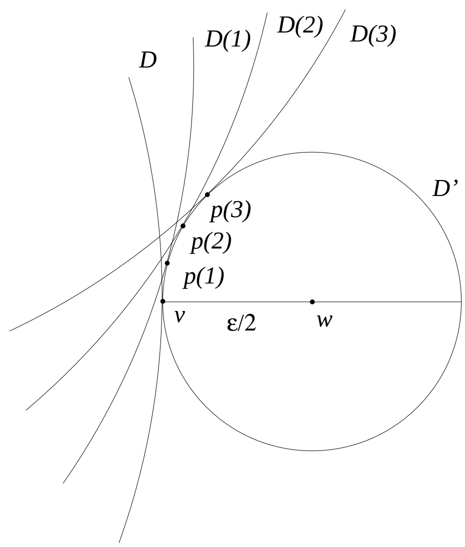

Figure 1. *Replace D by $D(1), D(2), \ldots, D(k)$.*

It remains to verify that the new realization $(P, \mathcal{D})$ meets the extra requirements stated in the lemma: it is in general position and each disk $D \in \mathcal{D}$ has a boundary point that does not belong to the closure of any other disks in $\mathcal{D}$. However, these conditions are automatically satisfied if δ is sufficiently small. For instance, each disk $D(i)$ has point on its boundary, very close to $p(i)$, which is not covered by any other disk in $\mathcal{D}$. To see that the same property holds for D', notice that *any* boundary point of D', "sufficiently far" from $p(1), p(2), \ldots, p(k)$, will do. This completes the induction step, and hence the proof of the lemma. □

Corollary 5.2. *For any $k \ge 2$, there exists a k-uniform hypergraph which is not two-colorable and which permits a planar realization by open disks.* □

6 Dual realization with axis-parallel rectangles

All rectangles in this section are assumed to be *closed*, but our results and proofs also apply to open rectangles.

Lemma 6.1. *For any rooted tree T, the hypergraph $H(T)$ permits a dual realization with axis-parallel rectangles.*

Proof: Let σ_0 and σ_1 denote the segments $y = x$, $1 \le x \le 2$ and $y = x + 2$, $0 \le x \le 1$. First, consider the sub-hypergraph H' of $H(T)$, consisting of all descendent hyperedges. We claim that it permits a dual realization with closed intervals and points of σ_1. To see this, choose an arbitrary interval in σ_1 to represent the root of T. If an interval I represents a vertex v of T and v has $k \ge 1$ children, choose any k pairwise disjoint sub-intervals of I to represent them. Finally, for every leaf v, pick any point of the interval representing v to represent the descendent hyperedge of H' that contains v. It is straightforward to check that the resulting system is indeed a dual realization of H'.

Now we construct a dual realization of $H(T)$ with axis-parallel rectangles. Let the descendent hyperedges be represented by the same point in σ_1 as in the construction above. For the sibling hyperedges, we choose distinct points of σ_0 to represent them. Let any vertex x of T be represented by the axis-parallel rectangle whose lower right corner is the point that represents the sibling hyperedge containing x, and whose intersection with σ_1 is the interval that represented x in the previous construction. (Note that the root of T is not contained in any sibling hyperedge. Therefore, if x is the root, we have to modify the above definition. In this case, let the lower right vertex of the corresponding rectangle be any point of σ_0 that does not represent any sibling hyperedge.) Clearly, the resulting system of points and rectangles is a dual representation of $H(T)$. $\square$

7 Planar and dual realizations with concave quadrilaterals

The aim of this section is to prove Theorem 2.8. For the proof, it is irrelevant whether we consider closed or open quadrilaterals.

One of the two diagonals of a concave quadrilateral Q is inside Q, the other is outside Q. We call the line of the diagonal outside Q the *supporting line of* Q.

Lemma 7.1 *For any rooted tree T and for any concave quadrilateral Q, the hypergraph $H(T)$ permits both planar and dual realizations with translates of Q. Moreover, we can achieve that all translates of Q used in the planar realization can be obtained from Q by translations parallel to its supporting line, while all points used in the dual realization lie on the supporting line.*

Proof: The two realizations are dual to each other, so it is enough to prove the existence of a *planar* realization. Let the vertices of Q be a, b, c, and d in this order, and assume b is the concave vertex. The supporting line of Q is the line ac. We start with a planar realization $(P, \mathcal{S})$, in which each member of $\mathcal{S}$ is a translate parallel to ac of one of the two infinite wedges W_a, W_c. Here the sides of W_a are the rays ad and ab, while the sides of the W_c are the rays cd and cb. Once we have such a planar realization, we can shrink the point set so that the wedges can be replaced by Q, without changing the containment relation.

In our planar realization, all sibling hyperedges will be represented by translates of W_a, while all descendent hyperedges will be represented by translates of

W_c. We construct the planar realization by induction on the depth of T, starting with the trivial case of depth 0.

For the inductive step, let v_0 be the root of T, let $v_1, \dots, v_k$ denote its children, and let T^i be the tree rooted at v_i, for $1 \leq i \leq k$. By the inductive hypothesis, for every i, $H(T^i)$ permits a *planar realization* $(P_i, \mathcal{S}_i)$, meeting the requirements. We assume that the following three additional conditions are also satisfied.

1. $W \cap P_j = \emptyset$, whenever $W \in S_i$ and $i \neq j$.
2. $P_i \cap W_a = \emptyset$, for all i.
3. For any i, there exists a point $x_i \in W_a$ such that, for any $W \in \mathcal{S}_j$, we have $x_i \in W$ if and only if $i = j$ and W is a translate of W_c.

To verify that one can make the above assumptions, note that $H(T^i)$ can also be realized by any translate of $(P_i, \mathcal{S}_i)$. Translating $(P_i, \mathcal{S}_i)$ through sufficiently fast increasing multiples of the vector ac, as i increases, makes all of the above three properties satisfied.

It is easy to see that one can find a point x, common to all translates of W_c in any of the families $\mathcal{S}_i$, with he property that x is not contained in W_a or in any of its translates considered. Let $y_i \in P_i$ denote the point representing the root v_i of T^i.

Now we are in a position to define the pair $(P, \mathcal{S})$ realizing T: let

$$P = ((\cup_i P_i) \cup \{x_i | 1 \leq i \leq k\} \cup \{x\}) \setminus \{y_i | 1 \leq i \leq k\},$$

and let $S = (\cup_i S_i) \cup \{W_a\}$. It is straightforward to check now that $(P, \mathcal{S})$ is a planar realization of $H(T)$, where sibling hyperedges are represented by translates of W_a parallel to the line ac and descendent hyperedges are represented by translates of W_c parallel to the same line. □

Proof of Theorem 2.8: Let Q be a concave quadrilateral and let $k \geq 1$ arbitrary. We need to show that not all k-fold coverings of the plane by translates of Q can be split into two coverings. Let us start with a dual realization $(P, \mathcal{S})$ of the k-uniform hypergraph $H_k = H(T_k)$ with translates of Q. We consider the set $\mathcal{S}'$ obtained from $\mathcal{S}$ by adding all translates of Q disjoint from P. Clearly, $\mathcal{S}'$ cannot be split into two covering, as every point of P can be covered only by members of $\mathcal{S}$, and we know that H_k is not two-edge-colorable.

It remains to check that $\mathcal{S}'$ is a k-fold covering of the plane. For this, we use the fact that the dual realization $(P, \mathcal{S})$ of H_k, whose existence is guaranteed by Lemma 7.1, satisfies that all points of P lie on the supporting line of Q. Clearly, any point that does not belong to this line is covered by infinitely many translates of Q that are disjoint from the line. For a point $r \notin P$ that belongs to the supporting we can still find infinitely many translates of Q which cover r and which are disjoint from the finite set P. If a is a vertex of Q on the supporting line then any translation that carries a point $a' \neq a$ of Q to r, where a' is sufficiently close to a, will do here. Finally, each point of P is covered by exactly k members of $\mathcal{S}$, as H_k is a k-uniform hypergraph. □

The proof of Lemma 7.1 applies not only to concave quadrilaterals, but to many other concave polygons Q', as well, implying that the families of translates of these polygons are not cover-decomposable. However, the statement is not true for *all* concave polygons. For instance, if Q' can be expressed as a finite union of translates of a given convex polygon, then the family of translates of Q' must be cover-decomposable. It would be interesting to find an exact criterion for deciding whether the family of translates of a polygon Q' is cover-decomposable.

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